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TWO LEMMAS THAT CHANGED GENERAL EQUILIBRIUM THEORY

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I would like to say first how I am grateful to David who was for me, alone in the beginning and later together with Sandra, a precious, elegant and always close and attentive friend.

I believe that I met David for the first time during the academic year 1981–82, a year of sabbatical leave he spent at the University of Paris-Dauphine. The dinner given by him in the gardens of Ecole Polytechnique before coming back to US is still a warm memory for all people who attended.

Bernard Cornet (researcher in Paris-Dauphine at this time) and I decided to spend the summer of 1982 in Berkeley at the invitation of Gérard Debreu. To David, we represented a little bit of the city of Paris he liked so much and he took good care of both of us and of our respective families. Besides seminars, I remember that we enjoyed jazz concerts in and around San Francisco, mathematical discussions based on the Möbius strips and other rudimentary exhibits he was constructing at this time or, simply, festivities in the family house on Hilldale Avenue, promenades with my daughter in the Science Museum, and ice cream parties in Berkeley with his (then teenaged) daughter Katharine.

It was the beginning of a long-lasting friendship that continued in Berkeley (several times, especially during the 1986 Special Year in Mathematical Economics organized by Gerard Debreu at the MSRI), at many conferences throughout the world, and also in Paris where David began to spend at least three months each year, after having bought an apartment in the Marais, not far from my own place. Thanks to his recommendation, my daughter later located her interior design agency in the same building. I knew by a mail of Sandra that he had passed away precisely when I was thinking that he would appear soon at rue de Sévigné.

From his giant scientific legacy to game theory and mathematical economics, I will speak only on subjects on which my comments may have some kind of legitimacy.

For mathematical economists and game theorists of my generation, the iconic papers he published alone or jointly with H.W. Kuhn and A.W. Tucker (and some other prestigious co-authors) in the *Princeton Annals of Mathematics Studies* (for example, Gale, 1956 and Gale et al., 1950) or in the *Cowles Commission Monographs* (for example, Gale et al., 1951) have been of a preeminent importance. This work and his book, Gale(1960), extended his international scientific fame to the USSR, where he visited in 1988-89. Together with a paper on dynamic economic models, Gale (1973), the book is still very popular and in use in the community working on optimal economic growth where he has a great number of intellectual descendants. I am one of a large number of scholars who have used Gale (1976) on the linear exchange model. The application to equilibrium in a discrete economy with money is still a source of inspiration for monetary theory.

Paper written in Honor of David Gale.

I would like to focus here on two lemmas that illustrate the seminal role played by David Gale in the development of the foundations of General Equilibrium Theory.

The first one, the Principal Lemma in "The law of supply and demand" (Gale, 1955), put the development of general-equilibrium existence proofs initially based on a simultaneous optimization approach due to Arrow–Debreu (1954) on a new trail. David Gale's paper points out the potential role of the properties (continuity, convexity, and Walras law) of the excess demand correspondence for achieving equilibrium, and proves under these conditions the existence of a free-disposal equilibrium for a bounded private ownership economy. The link between equilibrium existence and the Principal Lemma is of a great simplicity, but no more than was, in Arrow–Debreu's paper, the link between equilibrium of an economy and equilibrium of the associated abstract economy. In counterpart, the proof of the Principal Lemma is notably more elementary than the proof given in Debreu (1952) for his Theorem. While the Debreu proof for the existence of an equilibrium in a bounded abstract economy is based on the Eilenberg–Montgomery fixed point theorem, Gale proves the Principal Lemma first using the Knaster–Kuratowski–Mazurkiewicz lemma when the excess demand correspondence is a function, then extending the result to set-valued excess demand functions by a procedure similar to that used by Kakutani (1941) for extending Brouwer's theorem.

It is well-known that, except in very particular cases, an equilibrium existence theorem is based on (and equivalent to) some fixed point argument. Kuhn (1956) proposed a direct proof of the Principal Lemma by means of the stronger topological result of Eilenberg–Montgomery. Independently of Gale, but appreciated by Debreu, Nikaido (1956) proved a slight variant of the Principal Lemma based on the application of Kakutani's theorem to the product of the excess demand correspondences and a 'price manipulating function'. This interesting mixture of both excess demand and simultaneous optimization approaches was adopted by Debreu (1959) and established definitely the Principal Lemma as an elementary and useful lemma, still very popular among economists for the simplicity of the statement and its proof and the clear economic interpretation of the equilibrium mechanism. However, Debreu (1956) extended in a non-elementary way the lemma to the case where the cone of admissible prices is not no longer the positive cone, but instead is a convex cone that is not a linear subspace.[1]

In view of the respective roles of the three contributors and whatever the precise statement that is referred to, the lemma is typically referred to as the Gale–Nikaido–Debreu lemma.

Almost twenty years after the publication of "The law of supply and demand," the second lemma, jointly authored with A. Mas-Colell, reverses in some sense the direction stimulated by the first lemma to equilibrium existence proofs.

Formally, the fixed point Theorem (later referred to as Gale and Mas-Colell's lemma), stated in Section 2 of Gale and Mas-Colell (1975) and corrected in Gale and Mas-Colell (1979), gives a simpler and beautiful alternative to the complicated arguments given in Mas-Colell (1974). All previous equilibrium existence results (including in particular Negishi, 1960) depended on the hypothesis that consumers' preferences were represented by (convex) complete preorders on their consumption set. In most models, the existence of (continuous) quasi-concave utility functions was assumed fundamentally because transitivity and completeness of preference relations was used jointly with convexity arguments to show that the excess demand correspondence was convex valued. When consumption sets are convex and finite-dimensional, transitivity and completeness

[1] A variant of the result, Geistdoerfer-Florenzano (1982) has removed this unpleasant assumption.

of continuous preference relations is equivalent to the existence of a representation of preferences by well-behaved utility functions.

One may think that simultaneously discarding transitivity and completeness of preferences of agents was not a high priority objective for David Gale and more generally for game theorists. However, by replacing assumptions on preference preorders by assumptions on preference correspondences, it became clear that the simultaneous optimization approach could be used to create existence arguments that require the weakest assumptions and allow for all kind of externalities. As a matter of fact, this approach and the Gale and Mas-Colell fixed point theorem are used in a number of contexts where concepts and methods of general equilibrium are relevant. This kind of approach is even useful in proofs of the nonemptiness of the core of production economies without ordered preferences.

The proof of the fixed point theorem is unbelievably short, elegant, simple and elementary, provided one takes for granted that the finite dimensional case is covered by one of the Michael selection theorems, Theorem 3.1‴ in Michael (1956) (where the existence of a selection for a convex valued correspondence is obtained under mild conditions on the correspondence). The Gale and Mas-Colell paper brought to the attention of mathematical economists a deep mathematical result and revolutionized the research on existence of fixed points and maximal elements.

Neither the first nor second lemma have intellectual descendant in future Gale papers. Even if David was available for discussions, showing always a kindly attention, the task of developing applications for his results was in both cases left for others. Among the more than 90 papers that constitute his published work, and in addition to their long standing influence on the scientific area I am involved in, these two lemmas characterize for me David Gale's attitude toward Mathematics: a creative activity that makes hard demands in terms of rigor and culture, but always remains an elegant entertainment. David Gale will be remembered for the simplicity and beauty of his contributions in research areas of strong intellectual competition.

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